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CREEPING FLOW THROUGH CUBIC ARRAYS OF SPHERICAL BUBBLES

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Abstract—We consider creeping flow through a cubic array of identical spherical bubbles and compute the drag force exerted on a representative bubble in the array using a method originally employed by Hasimoto (1959) and recently modified by Sangani & Acrivos (1982). In addition to deriving analytic expressions for the drag to $O(c^{4})$, we present numerical results for the complete range of bubble volume fractions c for the three cubic arrays.

Dispersions of gases in liquids are widely employed in chemical reactors and it is often desirable to estimate the terminal velocity of a swarm of gas bubbles rising through a pool of liquid. This terminal velocity will in general depend on several variables, for example, the Reynolds number or the geometry, size and size distribution of the bubbles, so that no general theory seems possible. To date most theoretical studies on the subject have been restricted to the case of isolated bubbles rising in a fluid at rest except for the analysis of Wachholder (1973) and of Haber & Hetsroni (1981) who gave expressions for the $O(c)$ correction to the sedimentation velocity of a dilute emulsion of spherical drops, c being the volume fraction of the dispersed phase.

In order to determine the effect of particle–particle interactions at higher concentrations, we consider here the idealized case of an infinite periodic array of equal-sized bubbles of radius a^* rising through an incompressible Newtonian fluid of viscosity μ under conditions where the particle Reynolds number is sufficiently small so that the fluid motion satisfies the well known creeping flow equations. Since the array is assumed to remain periodic, the problem is equivalent to that of determining the force F exerted by the fluid moving with an average speed U on a representative bubble in an array of bubbles whose centers are fixed at positions given by

$$\mathbf{r_n} = h(n_1\mathbf{a}_{(1)} + n_2\mathbf{a}_{(2)} + n_3\mathbf{a}_{(3)}) \qquad (n_1, n_2, n_3 = 0, \pm 1, \pm 2, \cdots), \tag{1}$$

where $\mathbf{a}_{(1)}$, $\mathbf{a}_{(2)}$ and $\mathbf{a}_{(3)}$ are the basic vectors of the array and h is the characteristic dimension for the unit cell of the periodic array. Finally we assume that the viscous forces are much smaller compared to the forces due to the surface tension (i.e. $\mu U/\gamma \ll 1$) so that the bubbles remain spherical. Although the problem to be studied is quite artificial, it is hoped that its solution will offer valuable insight into the behavior of the physically realistic systems encountered in practice.

Recently, the present authors (1982) (henceforth referred to as I) calculated the force exerted by the fluid on the cubic arrays (simple, body-centered, and face-centered) of hard spheres using a modification of the method originally developed by Hasimoto (1959). In the present note this modified method will be applied to the case of cubic arrays of spherical bubbles.

As shown in I, the components of the velocity (non-dimensionalized by U) of the fluid are given by (the mean flow being in the x_1-direction)

$$u_1 = 1 + \frac{2B_{00}}{\tau_0} - \frac{1}{4\pi}\left\{\mathbf{G}\left(S_1 - \frac{\partial^2 S_2}{\partial x_1^2}\right) + \mathbf{H}\frac{\partial^2 S_1}{\partial x_1^2} - \mathbf{L}\left(\frac{\partial^4}{\partial x_2^4} - 6\frac{\partial^4}{\partial x_2^2\partial x_3^2} + \frac{\partial^4}{\partial x_3^4}\right)S_1\right\} \tag{2}$$

$$u_2 = \frac{1}{4\pi}\left\{\mathbf{G}\frac{\partial^2 S_2}{\partial x_1 \partial x_2} - \mathbf{H}\frac{\partial^2 S_1}{\partial x_1 \partial x_2} - \mathbf{L}\frac{\partial}{\partial x_1}\left(\frac{\partial^3}{\partial x_2^3} - 3\frac{\partial^3}{\partial x_2 \partial x_3^2}\right)\right\}S_1 \qquad [3]$$

$$u_3 = \frac{1}{4\pi}\left\{\mathbf{G}\frac{\partial^2 S_2}{\partial x_1 \partial x_3} - \mathbf{H}\frac{\partial^2 S_1}{\partial x_1 \partial x_3} - \mathbf{L}\frac{\partial}{\partial x_1}\left(\frac{\partial^3}{\partial x_3^3} - 3\frac{\partial^3}{\partial x_3 \partial x_2^2}\right)\right\}S_1 \qquad [4]$$

where the periodic functions S_1 and S_2 defined by Hasimoto (1959) have the following expansions in spherical harmonics near $r = 0$:

$$S_1 = \frac{1}{r} - \bar{c} + \frac{2\pi}{3\tau_0}r^2 + \sum_{n=2}^{\infty}\sum_{m=0}^{m\le n/2} a_{nm}Y_{2n}^{4m}(x_1, x_2, x_3) \qquad [5]$$

$$S_2 = \frac{r}{2} - \bar{c}_2 - \frac{\bar{c}}{6}r^2 + \frac{\pi r^4}{30\tau_0} + \sum_{n=2}^{\infty}\sum_{m=0}^{m\le n/2}(b_{nm} + \tilde{a}_{nm}r^2)Y_{2n}^{4m}(x_1, x_2, x_3) \qquad [6]$$

with

$$Y_n^m(x_1, x_2, x_3) = r^n P_n^m(\cos\theta)\cos m\phi, \qquad [7]$$

$$x_1 = r\cos\theta, \quad x_2 = r\sin\theta\cos\phi, \quad x_3 = r\sin\phi\sin\phi. \qquad [8]$$

Here $\bar{c}$, $\bar{c}_2$, τ_0, b_{nm}, a_{nm} and $\tilde{a}_{nm}$ are constants characteristic of the array. Their values for the simple, body-centered and face-centered cubic arrays have been reported in I. Also as defined in I,

$$\begin{Bmatrix}\mathbf{G}\\ \mathbf{H}\\ \mathbf{L}\end{Bmatrix} = \sum_{M=0}^{\infty}\sum_{m=0}^{m\le(1/2)M}\begin{Bmatrix}A_{nm}\\ B_{nm}\\ C_{nm}\end{Bmatrix}\left\{\frac{\partial^{2n}}{\partial x_1^{2n}}\left[\left(\frac{\partial}{\partial\xi}\right)^{4m} + \left(\frac{\partial}{\partial\eta}\right)^{4m}\right]\right\} \quad (M = n + 2m) \qquad [9]$$

with

$$\xi = x_2 + ix_3, \quad \eta = x_2 - ix_3,$$

and where the unknown coefficients A_{nm}, B_{nm}, and C_{nm} are to be determined by applying the boundary conditions described below.

The velocity components given by [2]–[4] automatically satisfy the periodic and cubic symmetry conditions (for the details of these boundary conditions the reader is referred to I) and therefore it remains to satisfy only the boundary conditions on the surface of any one bubble, specifically the one whose center coincides with the origin. Thus the boundary conditions at $r = a(= a^*/h)$ are that the normal component of the velocity and the tangential stresses are zero. In order to implement these boundary conditions for the purpose of determining the unknown coefficients in [9] we proceed as follows:

First of all, on substituting [5]–[9] into [2]–[4] we obtain

$$u_1 = \sum_{n=0}^{\infty}\sum_{m=0}^{m\le n/2} f_{1nm}(r)P_{2n}^{4m}(\cos\theta)\cos 4m\phi, \qquad [10]$$

$$u_2 = \sum_{n=1}^{\infty}\sum_{m=0}^{m\le n/2}\{f_{2nm}(r)P_{2n}^{4m-1}(\cos\theta)\cos(4m-1)\phi + f_{3nm}(r)P_{2n}^{4m+1}(\cos\theta)\cos(4m+1)\phi\},$$

$$(f_{2n0} \equiv 0, \quad f_{3nm} \equiv 0 \quad \text{if} \quad n = 2m) \qquad [11]$$

$$u_3 = \sum_{n=1}^{\infty} \sum_{n=0}^{m \le n/2} \{-f_{2nm}(r)P_{2n}^{4m-1}(\cos\theta)\sin(4m-1)\phi + f_{3nm}(r)P_{2n}^{4m+1}(\cos\theta)\cos(4m+1)\phi\} \tag{12}$$

where the functions $f_{inm}(r)$ $(i = 1, 2, 3)$ can be calculated as in I. Thus, for example,

$$f_{100} = 1 - \frac{1}{4\pi}\left\{\frac{4}{3r}\left[1 - \bar{c}r + \frac{2\pi}{3\tau_0}r^3\right]A_{00} - \frac{16\pi}{3\tau_0}B_{00}\right.$$

$$\left. + \left[\frac{16\pi}{15\tau_0} - 48b_{20} - 8a_{20}r^2\right]A_{10} + \cdots\right\}, \qquad [13]$$

$$f_{310} = \frac{1}{4\pi}\left\{\left[\frac{1}{3r} - 8\left(\frac{\pi}{45\tau_0} - b_{20}\right)r^2 + \frac{4}{7}a_{20}r^4\right]A_{00} + 2\left[\frac{1}{r^3} - 8r^2a_{20}\right]B_{00}\right.$$

$$\left. - \left[\frac{6}{7r^3} + 0(r^2)\right]A_{10} + \cdots\right\}. \qquad [14]$$

Next, on applying the boundary conditions and using various recursion relations among the spherical harmonics together with the orthogonality of these functions, we arrive at the following set of linear equations:

$$\left[(2n+4m)f_{1nm} + \frac{(4n+1)(2n-4m-1)}{(4n-3)}f_{1,n-1,m} + f_{2nm} - \frac{(4n+1)}{(4n-3)}f_{2,n-1,m}\right.$$

$$\left. - (2n+4m)(2n+4m+1)f_{3nm} + \frac{(4n+1)(2n-2-4m)(2n-1-4m)}{(4n-3)}f_{3,n-1,m}\right]_{r=a} = 0 \qquad [15]$$

$$\left(1 - a\frac{d}{dr}\right)\left\{-f_{2nm} + \frac{4n+1}{4n-3}f_{2,n-1,m} - (2m+4m)(2n+4m+1)f_{3nm}\right.$$

$$\left. + \frac{(4n+1)(2n-1-4m)(2n-2-4m)}{(4n-3)}f_{3,n-1,m}\right\}_{r=a} = 0 \qquad [16]$$

$$\left(1 - a\frac{d}{dr}\right)\left\{f_{1nm} - \frac{4n+1}{4n-3}f_{1,n-1,m} + \frac{1}{(2n+4m)}f_{2nm} + \frac{(4n+1)(2n-4m)}{(4n-3)(2n+4m)(2n+4m-1)}f_{2,n-1,m}\right.$$

$$\left. - (2n+4m+1)f_{3nm} - \frac{(4n+1)(2n-4m-2)}{(4n-3)}f_{3,n-1,m}\right\}_{r=a} = 0 \qquad n \ge 2m+1 \qquad [17]$$

$$\left(1 - a\frac{d}{dr}\right)\{4nf_{1nm} + f_{2nm}\}_{r=a} = 0, \quad n = 2m. \qquad [18]$$

The above set of infinite equations can be truncated to a finite set with an equal number of unknowns and the resulting equations can be solved in either of two ways (I). The first is a method of successive approximations which generates a series expansion in powers of a for each unknown while the second is a method of direct substitution in which the unknowns are determined by matrix inversion for a given value of a. Thus the method of successive approximation yields the expression for F:

$$K_b = \frac{F}{4\pi\mu Ua^*} = \left[1 - \frac{2}{3}\bar{c}a + \left(\frac{32\pi^2}{45\tau_0^2} - 240b_{20}^2\right)a^6 + 0(a^8)\right]^{-1}, \qquad [19]$$

where use has been made of the relationship

$$F = 2A_{00}\mu Uh. \tag{20}$$

In terms of the volume fraction c of the bubbles[19] can be rewritten as

$$K_b^{-1} = \begin{cases} 1 - 1.1734c^{1/3} - 0.1178c^2 + 0(c^{8/3}) & \text{(SC)} \\ 1 - 1.1946c^{1/3} + 0.3508c^2 + 0(c^{8/3}) & \text{(BCC)} \\ 1 - 1.1945c^{1/3} + 0.3611c^2 + 0(c^{8/3}) & \text{(FCC)} \end{cases} \tag{21}$$

We note that the coefficient of a in (19) is exactly 2/3 times the corresponding coefficient in the expression for the force on a cubic array of hard spheres (see [47] in I). This is to be expected from the method of reflections owing to the fact that the force on an isolated bubble is 2/3 times the force on an isolated hard sphere of the same radius, the viscosity and the velocity of the fluid at infinity being the same in each case.

The convergence-tested results obtained by employing the direct substitution method are given in table 1 in terms of a parameter χ defined by

$$\chi = \left(\frac{c}{c_{max}}\right)^{1/3}, \tag{22}$$

where c_{max} is the maximum volume fraction, i.e. the volume fraction of the bubbles when they are touching each other, whose value equals $\pi/6 = 0.5236$, $\sqrt{3}\pi/8 = 0.6802$, and $\sqrt{2}\pi/6 = 0.7405$, respectively for a simple, body-centered and face-centered cubic array. We see that these numerical results agree with[21] to within 5% for $\chi < 0.6$ for all the three cubic arrays. It is interesting to note that the ratio of the force experienced by a bubble in a closed packed configuration ($\chi = 1$) to that on an isolated bubble ($\chi = 0$) equals 12.8, 28.7, and 59, respectively, for a simple, body-centered, and face-centered cubic array, the corresponding values for the hard spheres being 42.1, 162.3, and 435.

Table 1. The dimensionless drag K_b for three cubic arrays

χ	K_b (SC)	K_b (BCC)	K_b (FCC)
0.1	1.104	1.117	1.121
0.2	1.233	1.266	1.276
0.3	1.396	1.460	1.480
0.4	1.609	1.724	1.760
0.5	1.898	2.101	2.168
0.6	2.315	2.678	2.808
0.7	2.967	3.650	3.927
0.8	4.11	5.54	6.26
0.85	5.06	7.29	8.59
0.9	6.51	10.26	13.01
0.95	8.84	15.99	23.34
0.97	10.18	19.86	31.95
0.99	11.8	25.3	47.0
1.0	12.8	28.7	59.0

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